Elizabeth Stephansen lecture:
Laplace eigenfunctions and elliptic PDE

Eugenia Malinnikova (Stanford University/NTNU)

Nasjonalt Matematikermøte 2022
Tromsø, September 1, 2022

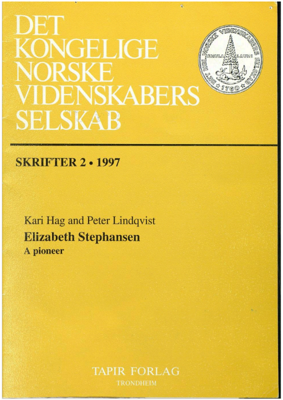
Elizabeth Stephansen (1872-1961)

- Born 10.03.1872 in Bergen
- Graduated from Eidgenössische Polytechnikum in Zurich (ETH) in 1896
- Obtained PhD in Mathematics (in absentia) from the University of Zurich in 1902
- Taught mathematics and physics at the Agricultural College of Norway (Norges Landbrukshøgskole på Ås)
- Was appointed Docent in Mathematics in 1921

Elizabeth Stephansen publications

Mary Ann Elizabeth Stephansen published four research articles on the theory of partial differential equations:
- Archiv for Mathematik og Naturvidenskap XXIV.4, 1902
- Archiv for Mathematik og Naturvidenskap XXV.6, 1903
- Prace Matematyczno-Fizyczne XVI, 1905
- Archiv for Mathematik og Naturvidenskap XXVII.6, 1906

Her work was influenced by Alf Guldberg.



Idun Reiten (b. 1942)

- Born 1.01.1942 in Klæbu
- Cand. real. (UiO) 1968
- PhD in Mathematics from the University of Illinois in 1971
- 1974-2012 Associate professor/ Professor at University in Trondheim/ NTNU
- Emmy Noether lecturer at ICM 2010
- Commander of the Order of St. Olav, 2014

Kari Hag (b. 1941)

- Born 4.04.1941 in Eidsvoll, Kari Jorun Blakkisrud
- Cand. mag. (Trondheim Lærerhøyskole) 1963, Cand. real. (UiO) 1967
- PhD in Mathematics from the University of Michigan in 1972
- 1972-2011 Associate professor/ Professor at NTH/NTNU
- Commander of the Order of St. Olav, 2018

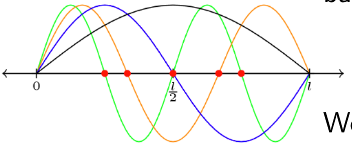
Lisa Lorentzen (b. 1943)

- PhD in Mathematics from NTH with Haakon Waadeland som veileder
- 1986-2013 Associate professor/ Professor at NTH/NTNU
- Author and co-author of a number of books/textbooks in mathematics

Berit Stensønes (1956-2022)

- PhD in Mathematics from Princeton University in 1985
- Professor at Michigan University
- 2013-2022 Professor at NTNU

Fourier basis



The Fourier basis on an interval is a basis of eigenfunctions of the operator

$$Lf = f''$$

We add the boundary condition $f(0) = f(l) = 0$, then

$$f_n = \sin(\pi n x/\ell), \quad Lf_n = -\lambda_n f_n,$$

$$\lambda_n = \pi^2 n^2 \ell^{-2}.$$

f_n has $n-1$ zeros in the interval $(0, l)$.

Dirichlet-Laplace eigenfunctions

We now consider a bounded domain $\Omega \subset \mathbb{R}^d$ and the Laplacian

$$\Delta u(x) = \sum_{j=1}^{d} \partial_j^2 u(x)$$

on the space of functions u vanishing on $\partial\Omega$. Then

$$\int_\Omega u \Delta u = - \int_\Omega |\nabla u|^2.$$

Dirichlet-Laplace eigenfunctions

We now consider a bounded domain $\Omega \subset \mathbb{R}^d$ and the Laplacian

$$\Delta u(x) = \sum_{j=1}^{d} \partial_j^2 u(x)$$

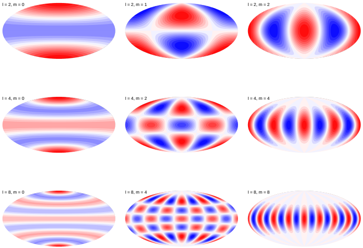
on the space of functions u vanishing on $\partial\Omega$. Then

$$\int_\Omega u \Delta u = - \int_\Omega |\nabla u|^2.$$

The operator $-\Delta$ has a discrete set of eigenvalues on this space, $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ and eigenfunctions ϕ_j such that

$$\Delta \phi_j + \lambda_j \phi_j = 0, \quad \phi_j = 0 \text{ on } \partial\Omega.$$

Eigenfunctions surfaces and compact Riemannian manifolds



On compact Riemannian manifolds: we consider the eigenfunctions of the Laplace-Beltrami operator φ_i:

$$\Delta\varphi_i = -\lambda_i\varphi, \; 0 = \lambda_1 < \lambda_2 \leq \ldots.$$

Standard examples are the sphere, where the eigenfunctions are the restrictions of homogeneous harmonic polynomials and the torus, where the eigenfunctions are (some) trigonometric polynomials.

Steklov eigenfunctions

Let Ω be a smooth bounded domain in $\mathbb{R}^d$. We consider the following problem:

$$\Delta u = 0 \text{ in } \Omega, \quad \partial_n u = \sigma u \text{ on } \partial\Omega.$$

This problem also has a discrete spectrum $0 = \sigma_1 < \sigma_2 \leq \sigma_3 \leq \dots$. These are eigenvalues of the (non-local) Dirichlet-to-Neumann map.

Steklov eigenfunctions

Let Ω be a smooth bounded domain in $\mathbb{R}^d$. We consider the following problem:

$$\Delta u = 0 \text{ in } \Omega, \quad \partial_n u = \sigma u \text{ on } \partial\Omega.$$

This problem also has a discrete spectrum $0 = \sigma_1 < \sigma_2 \leq \sigma_3 \leq \ldots$. These are eigenvalues of the (non-local) Dirichlet-to-Neumann map.
For the case of $\Omega = B$, the corresponding eigenfunctions are homogeneous harmonic polynomials.

Yau's conjecture on zero set of eigenfunctions

Let ϕ_λ be an eigenfunction with eigenvalue λ. We consider its zero set

$$Z(\phi_\lambda) = \{x : \phi_\lambda(x) = 0\}.$$

Yau's conjecture on zero set of eigenfunctions

Let ϕ_λ be an eigenfunction with eigenvalue λ. We consider its zero set

$$Z(\phi_\lambda) = \{x : \phi_\lambda(x) = 0\}.$$

In the 1970s Yau conjectured that (for large λ)

$$c\sqrt{\lambda} \leq \mathcal{H}^{d-1}(Z(\phi_\lambda)) \leq C\sqrt{\lambda}.$$

Yau's conjecture on zero set of eigenfunctions

Let ϕ_λ be an eigenfunction with eigenvalue λ. We consider its zero set

$$Z(\phi_\lambda) = \{x : \phi_\lambda(x) = 0\}.$$

In the 1970s Yau conjectured that (for large λ)

$$c\sqrt{\lambda} \leq \mathcal{H}^{d-1}(Z(\phi_\lambda)) \leq C\sqrt{\lambda}.$$

The conjecture was proved in 1988 by Donnelly and Fefferman for the case of real analytic manifold and a domain in $\mathbb{R}^d$ with real analytic boundary. In 2017 Logunov proved the lower bound. The upper bound is still open for general manifolds.

Yau's conjecture on zero set of eigenfunctions

Let ϕ_λ be an eigenfunction with eigenvalue λ. We consider its zero set

$$Z(\phi_\lambda) = \{x : \phi_\lambda(x) = 0\}.$$

In the 1970s Yau conjectured that (for large λ)

$$c\sqrt{\lambda} \leq \mathcal{H}^{d-1}(Z(\phi_\lambda)) \leq C\sqrt{\lambda}.$$

The conjecture was proved in 1988 by Donnelly and Fefferman for the case of real analytic manifold and a domain in $\mathbb{R}^d$ with real analytic boundary. In 2017 Logunov proved the lower bound. The upper bound is still open for general manifolds.
For the Steklov eigenfunctions the corresponding conjecture is that for an eigenfunction u_σ we have

$$c\sigma \leq \mathcal{H}^{d-1}(Z(u_\sigma)) \leq C\sigma.$$

This is open. Partial results are due to Decio.

Growth of Laplace eigenfunctions on compact manifolds

Donnelly and Fefferman proved the following growth estimate for Laplace eigenfunctions on compact Riemannian manifolds:

$$\Delta\phi_\lambda + \lambda\phi_\lambda = 0$$

For any geodesic ball $B_r(x)$

$$\max_{B_{2r}(x)} |\phi_\lambda| \leq \exp(C\sqrt{\lambda}) \max_{B_r(x)} |\phi_\lambda|,$$

where r is assumed to be small enough.
In particular the vanishing order of an eigenfunction at any point of Ω is bounded by $C\sqrt{\lambda}$. Similar result for Steklov eigenfunctions was obtained by Zhu.

Growth of Laplace eigenfunctions on compact manifolds

Donnelly and Fefferman proved the following growth estimate for Laplace eigenfunctions on compact Riemannian manifolds:

$$\Delta\phi_\lambda + \lambda\phi_\lambda = 0$$

For any geodesic ball $B_r(x)$

$$\max_{B_{2r}(x)} |\phi_\lambda| \leq \exp(C\sqrt{\lambda}) \max_{B_r(x)} |\phi_\lambda|,$$

where r is assumed to be small enough.
In particular the vanishing order of an eigenfunction at any point of Ω is bounded by $C\sqrt{\lambda}$. Similar result for Steklov eigenfunctions was obtained by Zhu.

An eigenfunction corresponding to eigenvalue λ (σ) behaves as a polynomial of degree $\sqrt{\lambda}$ (σ).

Bernstein's and Markov inequalities for polynomials

Theorem (Bernstein)

If p_n is a trigonometric polynomial of degree n then $\max_{\mathbb{T}} |p_n'| \leq n \max_{\mathbb{T}} |p_n|$.

Theorem (Markov)

If P_n is an algebraic polynomial of degree n then $\max_{[-1,1]} |P_n'| \leq n^2 \max_{[-1,1]} |P_n|$.

Bernstein's and Markov inequalities for polynomials

Theorem (Bernstein)

If p_n is a trigonometric polynomial of degree n then $\max_{\mathbb{T}} |p_n'| \leq n \max_{\mathbb{T}} |p_n|$.

Theorem (Markov)

If P_n is an algebraic polynomial of degree n then $\max_{[-1,1]} |P_n'| \leq n^2 \max_{[-1,1]} |P_n|$.

Markov's inequality holds in higher dimensions. For harmonic polynomials the result is better than for general algebraic polynomials:

Theorem (Szegö (1940), Marzo (2007))

Let H_n be a harmonic polynomial of degree n in $\mathbb{R}^d$

$$\sup_B |\nabla H_n| \leq Cn \sup_B |H_n|$$

where B is the unit ball in $\mathbb{R}^d$.

Bernstein's inequality for eigenfunctions

Let Ω be a smooth subdomain of $\mathbb{R}^d$, there exists $C = C(\Omega)$ such that for any eigenfunction ϕ_λ of the Laplace -Beltrami operator, we have

$$\sup_\Omega |\nabla \phi_\lambda| \leq C\sqrt{\lambda} \sup_\Omega |\phi_\lambda|.$$

Moreover, a similar inequality holds for linear combinations of eigenfunctions.

Theorem (Filbir, Mhaskar, 2010)

Let $\phi = \sum_{\lambda_j \leq \lambda} c_j \phi_{\lambda_j}$ *then* $\sup_\Omega |\nabla \phi| \leq C\sqrt{\lambda} \sup_\Omega |\phi|$.

Markov's inequality for eigenfunctions

Donnelly and Fefferman (1990) showed for any ball $B(x, r)$ in Ω

$$\|\nabla\phi_\lambda\|_{L^2(B)} \le C\lambda^{1/2} r^{-1} \|\phi_\lambda\|_{L^2(B)}, \quad \text{and}$$

$$\|\nabla\phi_\lambda\|_{L^\infty(B)} \le C\lambda^{(d+2)/2} r^{-1} \|\phi_\lambda\|_{L^\infty(B)}$$

and conjectured that the coefficient in the last inequality can be improved to $C\lambda^{1/2} r^{-1}$.

Markov's inequality for eigenfunctions

Donnelly and Fefferman (1990) showed for any ball $B(x, r)$ in Ω

$$\|\nabla \phi_\lambda\|_{L^2(B)} \le C\lambda^{1/2} r^{-1} \|\phi_\lambda\|_{L^2(B)}, \quad \text{and}$$

$$\|\nabla \phi_\lambda\|_{L^\infty(B)} \le C\lambda^{(d+2)/2} r^{-1} \|\phi_\lambda\|_{L^\infty(B)}$$

and conjectured that the coefficient in the last inequality can be improved to $C\lambda^{1/2} r^{-1}$.

Theorem (Decio, M, 2022)

There is $C = C(\Omega)$ such that for any eigenfunction ϕ_λ with $\Delta \phi_\lambda + \lambda \phi_\lambda = 0$ and any ball $B(x, r) \subset \Omega$ with $r < C\lambda^{-1/2}$ and any $\delta > 0$, we have

$$\sup_{B(x,r)} |\nabla \phi_\lambda| \le C_\delta \frac{\sqrt{\lambda}(\log \lambda)^{2+\delta}}{r} \sup_{B(x,r)} |\phi_\lambda|.$$

Similarly, for Steklov eigenfunctions $\partial_n u_\sigma = \sigma u$ on $\partial\Omega$, we have

$$\sup_{B(x,r)} |\nabla u_\sigma| \le C_\delta \frac{\sigma(\log \sigma)^{2+\delta}}{r} \sup_{B(x,r)} |u_\sigma|.$$

From Laplace eigenfunctions to harmonic functions

$$\Delta\phi_\lambda + \lambda\phi_\lambda = 0 \quad \text{vs} \quad \Delta u = 0.$$

Let ϕ_λ satisfy $\Delta\phi_\lambda + \lambda\phi_\lambda = 0$ on Ω.

Old trick: define a harmonic function u on $\Omega \times \mathbb{R}$ by

$$u(x,t) = \phi_\lambda(x)\exp(\sqrt{\lambda}t),$$

From Laplace eigenfunctions to harmonic functions

$$\Delta\phi_\lambda + \lambda\phi_\lambda = 0 \quad \text{vs} \quad \Delta u = 0.$$

Let ϕ_λ satisfy $\Delta\phi_\lambda + \lambda\phi_\lambda = 0$ on Ω.

Old trick: define a harmonic function u on $\Omega \times \mathbb{R}$ by

$$u(x,t) = \phi_\lambda(x)\exp(\sqrt{\lambda}t),$$

The following inequality for the doubling index holds for u,

$$\log \frac{\max_{B_{2r}(x,t)}|u|}{\max_{B_r(x,t)}|u|} \le C\sqrt{\lambda},$$

for $r < r_0$.

Density of the zero set for Laplace eigenfunctions

Suppose that ϕ_λ is positive on a ball $B(x, 2r) \subset \Omega$. Then the lifted function $u(x, t)$ is also positive in the corresponding ball $\tilde{B}((x, 0), 2r) \subset \omega \times \mathbb{R}$.

Density of the zero set for Laplace eigenfunctions

Suppose that ϕ_λ is positive on a ball $B(x, 2r) \subset \Omega$. Then the lifted function $u(x, t)$ is also positive in the corresponding ball $\tilde{B}((x, 0), 2r) \subset \omega \times \mathbb{R}$. Apply the Harnack inequality for $\tilde{B}(x, r)$ we conclude

$$u(x) \geq \inf_{\tilde{B}((x,0),r)} u \geq c \sup_{\tilde{B}((x,0),r)} \geq c e^{r\sqrt{\lambda}} u(x)$$

Thus $r \leq C_1/\sqrt{\lambda}$ and the zero set of ϕ_λ is $2C_1/\sqrt{\lambda}$ dense.

Density of the zero set for Laplace eigenfunctions

Suppose that ϕ_λ is positive on a ball $B(x,2r)\subset\Omega$. Then the lifted function $u(x,t)$ is also positive in the corresponding ball $\tilde{B}((x,0),2r)\subset\omega\times\mathbb{R}$. Apply the Harnack inequality for $\tilde{B}(x,r)$ we conclude

$$u(x)\geq \inf_{\tilde{B}((x,0),r)} u \geq c \sup_{\tilde{B}((x,0),r)} \geq ce^{r\sqrt{\lambda}}u(x)$$

Thus $r\leq C_1/\sqrt{\lambda}$ and the zero set of ϕ_λ is $2C_1/\sqrt{\lambda}$ dense.

Surprisingly, for Steklov eigenfunctions the statement is wrong. Recently Bruno and Galkowski showed that there is an analytic domain Ω and a subdomain Ω_1 on which a sequence of Stekolv eigenfunctions does not vanish.

The frequency function

$$\operatorname{div}(A\nabla u) = 0, \quad A \text{ is symmetric, elliptic and Lipschitz}$$

For simplicity let $A = I$ and define

$$H_u(x,r) = |\partial B_r|^{-1} \int_{\partial B_r(x)} |u|^2, \quad F_u(x,r) = \frac{rH'(r)}{H(r)}.$$

F_u is called the frequency function of u.

- If p is a homogeneous polynomial of degree m then $F_p(0,r) = 2m$.

The frequency function

$$\mathrm{div}(A\nabla u) = 0, \quad A \text{ is symmetric, elliptic and Lipschitz}$$

For simplicity let $A = I$ and define

$$H_u(x,r) = |\partial B_r|^{-1} \int_{\partial B_r(x)} |u|^2, \quad F_u(x,r) = \frac{rH'(r)}{H(r)}.$$

F_u is called the frequency function of u.

- If p is a homogeneous polynomial of degree m then $F_p(0,r) = 2m$.
- $\lim_{r\to 0} F_u(x,r)$ equals to (two times) the order of vanishing of u at x.

The frequency function

$$\mathrm{div}(A\nabla u) = 0, \quad A \text{ is symmetric, elliptic and Lipschitz}$$

For simplicity let $A = I$ and define

$$H_u(x,r) = |\partial B_r|^{-1} \int_{\partial B_r(x)} |u|^2, \quad F_u(x,r) = \frac{rH'(r)}{H(r)}.$$

F_u is called the frequency function of u.

- If p is a homogeneous polynomial of degree m then $F_p(0,r) = 2m$.
- $\lim_{r\to 0} F_u(x,r)$ equals to (two times) the order of vanishing of u at x.
- Doubling index versus frequency function

$$cF_u(x,r) - C \leq \log \frac{\max_{B(x,2r)} |u|}{\max_{B(x,r)} |u|} \leq CF_u(x,4r) + C.$$

Monotonicity of the frequency function

Monotonicity of Almgen's frequency function (Garofalo-Lin, 1986).

$$H_u(x,r) = |\partial B_r|^{-1} \int_{\partial B_r(x)} |u|^2, \quad F_u(x,r) = \frac{rH'(r)}{H(r)}.$$

If $\Delta u = 0$ then $F_u(x,r)$ is non-decreasing in r.
For more general elliptic equation $F_u(x,r)e^{cr}$ is non-decreasing.

Monotonicity of the frequency function

Monotonicity of Almgen's frequency function (Garofalo-Lin, 1986).

$$H_u(x,r) = |\partial B_r|^{-1} \int_{\partial B_r(x)} |u|^2, \quad F_u(x,r) = \frac{rH'(r)}{H(r)}.$$

If $\Delta u = 0$ then $F_u(x,r)$ is non-decreasing in r.
For more general elliptic equation $F_u(x,r)e^{cr}$ is non-decreasing.

- The monotonicity result implies the doubling property of the measure $|u|^2$.
- Using the doubling and the Caccioppoli inequality, one can obtain the reverse Hölder inequality (RH) for $|u|^2$.
- RH means that $|u| \in A_\infty$ and equivalently $\log |u| \in BMO$.

Markov's inequality for eigenfunctions: scheme

Main steps of the proof:

- Harmonic polynomials of degree n:
  $\sup_{B(x,r)} |\nabla P| \leq nr^{-1} \sup_{B(x,r)} |P|$.
- Harmonic functions: $\sup_{B(x,r)} |\nabla h| \leq Nr^{-1} \sup_{B(x,r)} |h|$, where $N = N_h(x, 2r)$.
  Idea: approximate harmonic function in $B(x, r)$ by a polynomial of degree $5N$.
- Steklov eigenfunctions and Dirichlet-Laplace eigenfunctions using the lifting trick.
- Solutions to elliptic PDEs
  $\sup_{B(x,r)} |\nabla u| \leq N(\log N)^{2+\delta} r^{-1} \sup_{B(x,r)} |h|$, where $N = N_u(x, R)$, $r < R/N$.
  Idea: approximate a solution to elliptic equation by a solution of elliptic equation with analytic coefficients with the same boundary data, they have comparable frequencies!

Thank You for your attention